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Control charts —

Part 7: **Multivariate control charts**

Cartes de contrôle —

Partie 7: Cartes de contrôle multivariées

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Foreword

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This document was prepared by Technical Committee ISO/TC 69, *Applications of statistical methods*, Subcommittee SC 4, *Applications of statistical methods in process management*.

A list of all parts in the ISO 7870 series can be found on the ISO website.

Any feedback or questions on this document should be directed to the user's national standards body. A complete listing of these bodies can be found at www.iso.org/members.html.

Introduction

When a number of quality characteristics are to be controlled simultaneously, the usual practice has been to maintain a separate (univariate) chart for each characteristic. Unfortunately, this can give misleading results when the characteristics are highly correlated. Process monitoring of problems in which several related variables are of interest are collectively known as multivariate statistical process control (MSPC). The most useful tools of multivariate statistical process control are multivariate control charts. Multivariate control charts are applied for statistical process evaluation and control under the consideration of dependability between quality characteristics.

The function of a multivariate statistical process control system is to provide a statistical signal when assignable causes of variation are present. The systematic elimination of assignable causes of excessive variation, through continuous determined efforts, brings the process into a state of statistical control. Once the process is operating in statistical control, its performance is predictable and its capability to meet the specifications can then be assessed.

The main purpose of this document is to show how multivariate control charts can be used for process control in terms of SPC and how the state of process stability can be assessed in a multivariate way. ISO 22514-6 provides a calculation method for capability statistics for process parameters or product characteristics following a multivariate normal distribution or approximately multivariate normal.

Multivariate charts are based on multivariate characteristics where more than one characteristic is to be monitored in connection with others. In practice, a multivariate control chart is always applied with the support of software, such as Minitab, JMP, and Q-DAS[1].

1) MINITAB is the trade name of a product supplied by Minitab Inc. JMP is the trade name of a product supplied by SAS Institute Inc. Q-DAS is the trade name of a product supplied by Q-DAS GmbH. This information is given for the convenience of users of this document and does not constitute an endorsement by ISO of these products.

Control charts —

Part 7: Multivariate control charts

1 Scope

This document describes the construction and use of multivariate control charts in statistical process control (SPC) and establishes methods for using and understanding this generalized approach to control charts where the characteristics being measured are from variables data.

The use of principal component analysis (PCA) and partial least squares (PLS) in the field of multivariate statistical process control is not presented in this document

NOTE The document describes the current state of the art of multivariate control charts that are being applied in practice nowadays. It does not describe the current state of scientific research on the topic.

2 Normative references

The following documents are referred to in the text in such a way that some or all of their content constitutes requirements of this document. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

ISO 3534-2, *Statistics — Vocabulary and symbols — Part 2: Applied statistics*

3 Terms and definitions

For the purposes of this document, the terms and definitions given in ISO 3534-2 apply.

ISO and IEC maintain terminological databases for use in standardization at the following addresses:

— ISO Online browsing platform: available at https://www.iso.org/obp

— IEC Electropedia: available at http://www.electropedia.org/

3.1
multivariate characteristics
multivariate quantity where the set of features consists of d quantities that are alone or combined with the quality of a product

Note 1 to entry: Following ISO 7870-2, these quantities are denoted as quality characteristics X_i where $i = 1, 2, ...,d$.

Note 2 to entry: The observation of multivariate characteristics can be expressed as the vector $\mathbf{x}=(x_1, x_2, ..., x_d)^T$. Thus, a multivariate quantity can be considered as a feature vector of a product. The value of the multivariate quantity is represented by a point in the d-dimensional feature space.

Note 3 to entry: All single quantities combined in the multivariate vector can be measured in the same product or object.

Note 4 to entry: If the multivariate quantity is described by means of statistics, the vector is considered as a random vector following a d-dimensional multivariate distribution.

3.2
confidence region
d-dimensional region for a multivariate characteristics of *d*-dimension and defined for a specified confidence level

Note 1 to entry: The region is limited by lines, surfaces or hyper-surfaces in the *d*-dimensional space.

Note 2 to entry: Form and size of the region are defined by one or more parameters.

4 Abbreviated terms and symbols

4.1 Abbreviated terms

SPC	statistical process control
MSPC	multivariate statistical process control
PCA	principal component analysis
PLS	partial least squares
UCL	upper control limit
LCL	lower control limit
ARL	average run length
EWMA	exponential weighted moving average
MEWMA	multivariate exponential weighted moving average

4.2 Symbols

$B_{1-\alpha,\nu_1,\nu_2}$	the $1-\alpha$ quantile of beta distribution with degree of freedom ν_1 and ν_2
d	number of dimensions for multivariate characteristics
D_j^2	the statistic plotted of a phase IIχ^2 control chart
$E(\lvert\mathbf{S}\rvert)$	mean of $\lvert\mathbf{S}\rvert$
$F_{1-\alpha,\nu_1,\nu_2}$	the $1-\alpha$ quantile of F distribution with degree of freedom ν_1 and ν_2
h	upper control limit of MEWMA control chart
L_{CL}	lower control limit
m	number of subgroups
n	size of each subgroup
$\mathrm{N}_d(\boldsymbol{\mu},\boldsymbol{\Sigma})$	d-dimensional normal distribution with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
s_{ab}	covariance between the a-th and b-th quality characteristics with n=1
s_{abj}	covariance between the a-th and b-th quality characteristics in the j-th subgroup with n>1
s_i^2	variance of the i-th quality characteristic with n=1

s_{ij}^2	variance of the i-th quality characteristic in the j-th subgroup with n>1
$\bar{s}_i^2$	average of s_{ij}^2 over all m subgroups for the i-th quality characteristic with n>1
$\bar{s}_{ab}$	average of s_{abj} over all m subgroups for the covariance between the a-th and b-th quality characteristics with n>1
$\mathbf{S}$	sample variance-covariance matrix with n=1
$\bar{S}$	sample variance-covariance matrix with n>1
$\lvert\mathbf{S}\rvert$	determinant of the sample variance-covariance matrix **S**
T_j^2	the statistic plotted of a phase I T^2-chart.
T_f^2	the statistic plotted of a phase II T^2-chart
tr	trace operator
U_{CL}	upper control limit
$V(\lvert\mathbf{S}\rvert)$	variance of $\lvert\mathbf{S}\rvert$
x_{ij}	the j-th observation on the i-th quality characteristic with n=1
x_{ijk}	the k-th observation in the j-th subgroup on the i-th quality characteristic with n>1
$\bar{x}_{ij}$	mean of the i-th quality characteristic in the j-th subgroup with n>1
$\bar{\bar{x}}_i$	average of $\bar{x}_{ij}$ over all m subgroups for the i-th quality characteristic with n>1
$\mathbf{x}$	an observation vector
$\mathbf{x}_j$	vector of j-th observation with n=1
$\mathbf{x}_f$	vector of a future individual observation with n=1
$\bar{\mathbf{x}}$	sample mean vector with n=1
$\bar{\mathbf{x}}_j$	mean of the j-th rational subgroup with n>1
$\bar{\mathbf{x}}_f$	mean of a future rational subgroup with n>1
$\bar{\bar{\mathbf{x}}}$	sample mean vector with n>1
$\{\bar{\bar{x}}_i\}$	i-th element of the vector $\bar{\bar{\mathbf{x}}}$
Y_j^2	the statistic plotted of MEWMA control chart.
$\mathbf{Z}_j$	MEWMA statistic
$\chi_{1-\alpha,\nu}^2$	the $1-\alpha$ quantile of χ^2 distribution with degree of freedom ν
δ	shift size of the mean vector
$\boldsymbol{\lambda}$	MEWMA moving parameter vector
λ	EWMA moving parameter, $0 < \lambda \leq 1$
$\boldsymbol{\mu}$	mean vector of multivariate characteristics

$\boldsymbol{\mu}_0$ pre-specified mean vector of multivariate characteristics

ρ_{y_1, y_2} correlation coefficient between y_1 and y_2

$\boldsymbol{\Sigma}$ variance-covariance matrix of multivariate characteristics

Σ_0 pre-specified variance-covariance matrix of multivariate characteristics

Σ_{Z_j} variance-covariance matrix of MEWMA statistic $\mathbf{Z}_j$

$(\cdot)^{-1}$ inverse operator

$(\cdot)^{\mathrm{T}}$ transpose operator

5 Purpose and classification of multivariate control charts

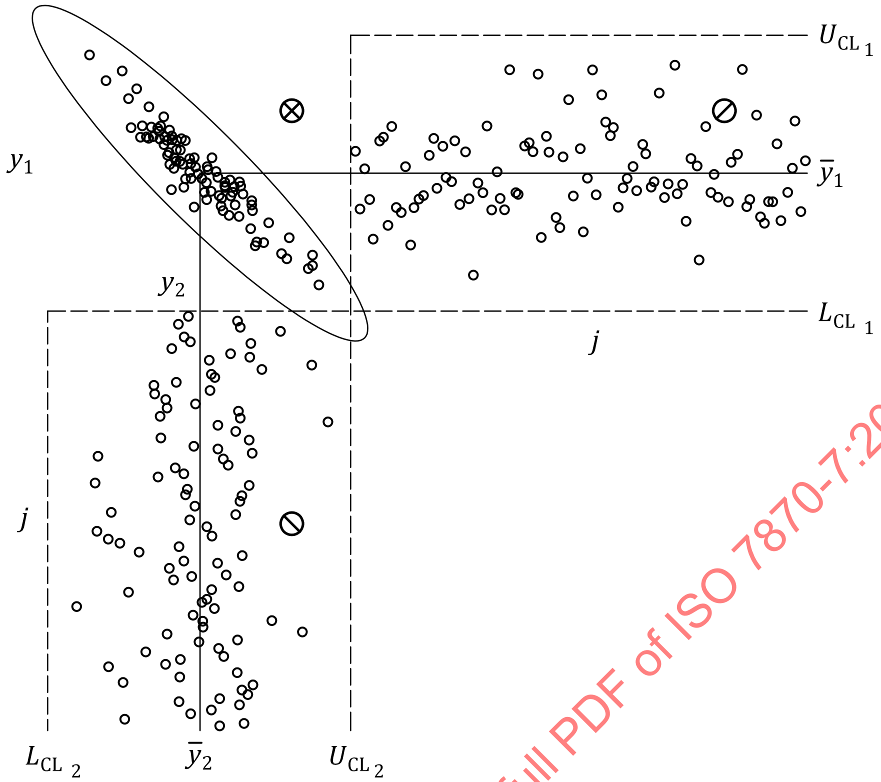
5.1 Purpose and applying conditions for multivariate control charts

There are many situations in which the simultaneous monitoring or control of two or more related quality characteristics is necessary. The difficulty with using independent univariate control charts is illustrated in Figure 1. Only two quality characteristics (y_1, y_2) are considered for ease of illustration.

Suppose that, when the process is in a state of statistical control where only common cause variation is present, both y_1 and y_2 follow a normal distribution but are correlated (ρ_{y_1, y_2} = −0,94) as illustrated in the joint plot of y_1 vs. y_2 in Figure 1. The ellipse represents a contour for the in-control process, with 0,997 3-quantile, corresponding to risk of a false alarm of 0,002 7 in the Shewhart chart, and the points represent a set of individual observations from this distribution. The same observations are also plotted in Figure 1 as individual Shewhart control charts on y_1 and y_2 vs. the observation number (time) with their corresponding upper and lower control limits (the 0,998 65-quantiles).

By looking at each of the individual Shewhart control charts, the process appears to be clearly in a state of statistical control, and none of the points give any indication of a problem. The true situation is only revealed in the bivariate y_1 vs. y_2 plot where it is seen that the lot of product indicated by the $\otimes$ is clearly outside the confidence region and is clearly different from the normal “in-control” population of the product.

If the quality characteristics are not independent, which usually would be the case if they relate to the same product, there is no easy way to measure the distortion in the joint control procedure. Process-monitoring problems in which several related variables are of interest are sometimes called multivariate quality control problems. This subject is particularly important, as automatic inspection procedures make it relatively easy to measure many parameters on each unit of product manufactured. For example, many chemical and process plants and semi-conductor manufacturers routinely maintain manufacturing databases with the process and quality data on hundreds of variables. Monitoring or analysing these data with univariate SPC procedures is often ineffective. Multivariate control charts are applied for statistical process evaluation and control under consideration of dependability between the product or process characteristics.



Key

$\bar{y}_1$	mean of y_1	U_{CL_2}	upper control limit of y_2
$\bar{y}_2$	mean of y_2	L_{CL_2}	lower control limit of y_2
j	observation number	⊘	point corresponding to y_1
U_{CL_1}	upper control limit of y_1	⃠	point corresponding to y_2
L_{CL_1}	lower control limit of y_1	⊗	point corresponding to (y_1, y_2)

Figure 1 — Quality control of two variables

Multivariate control charts work well when the number of process variables is not too large – ten or fewer. As the number of variables grows, however, traditional multivariate control charts lose efficiency with regard to shift detection. A popular approach in these situations is to reduce the dimensionality of the problem. This can be done with the use of projection methods such as principal component analysis (PCA) or partial least squares (PLS). These two methods are based on building a model from a historical data set, that is assumed to be in control. After the model has been built, a future observation is checked as to whether it fits well or not in the model.

In the SPC univariate case, the normal distribution is generally used to describe the behaviour of a continuous quality characteristic. The same approach can be used in the multivariate case. A multivariate normal distribution is applied as the basic assumption for a multivariate characteristics.

5.2 Classification of multivariate control charts

If the multivariate characteristics is considered to be a random vector with a multivariate normal distribution, this distribution is characterized by a mean vector **μ** and a variance-covariance matrix **Σ** (see Annex C). Obviously from the viewpoint of the application of multivariate process control, multivariate control charts can be applied to monitor the mean shift and process dispersion separately. Thus, for the application, multivariate control charts can be classified as follows:

a) multivariate control charts for mean shift;

b) multivariate control charts for process dispersion.

For the mean shift, multivariate control charts with unweighted averages are analogous to the Shewhart $\bar{X}$ chart or chart for individuals. They use information only from the current sample and are relatively insensitive to small and moderate shifts in the mean vector. Multivariate control charts with weighted averages such as multivariate EWMA control chart can be used to overcome this problem. Just like EWMA charts are generally used for detecting small shifts in the process mean and they usually detect shifts of 0,5 sigma to 2 sigma much faster. Thus, multivariate control charts for mean shift can be classified as follows:

i) multivariate control charts with unweighted averages (see Clause 6), such as χ^2 and T^2 chart;

ii) multivariate control charts with weighted averages (see Clause 7), such as multivariate EWMA control chart.

Figure 2 is given to show how to select multivariate control charts.

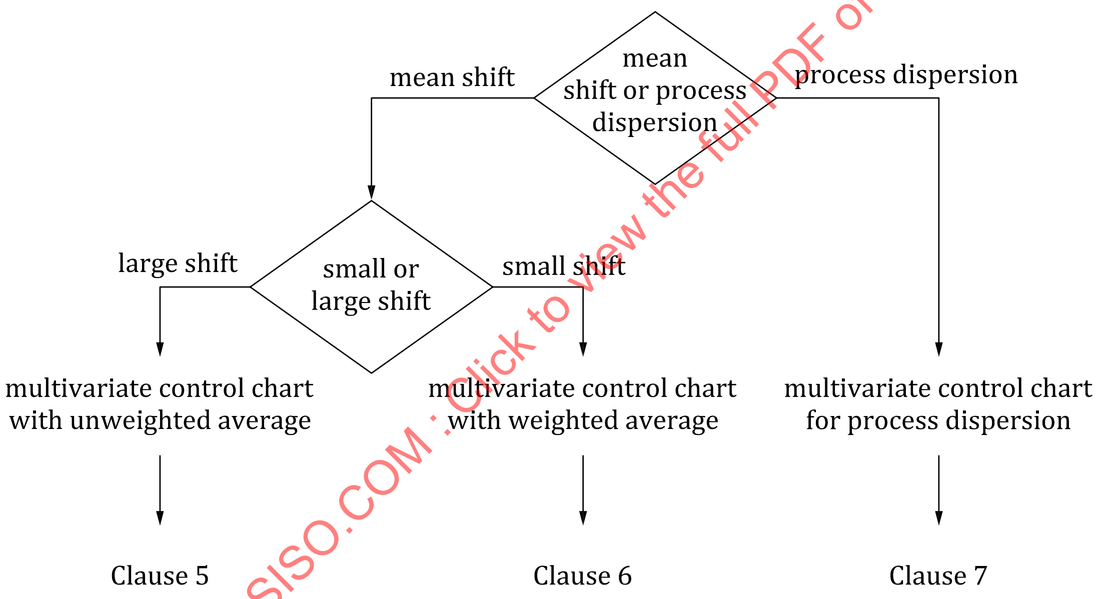


Figure 2 — Multivariate control chart selection flow chart

6 Multivariate control charts with unweighted averages for mean shift

6.1 General

For each of the multivariate control charts, there are two distinct situations:

a) when no pre-specified process parameter values are given, and

b) when pre-specified process parameter values are given.

The pre-specified or known process parameter values can be defined by target values or by requirements or by estimated values that have been determined by the data under the condition of a process in control.

There are two distinct phases of control charting practice.

i) Phase I: control charts are used for retrospective testing of whether the process was in-control when the first subgroups were being drawn. Once this is accomplished, the control chart is used to define what is meant by a process being statistically in-control. This is referred to as the retrospective use of control charts;

ii) Phase II: control charts are used for testing whether the process remains in-control when future subgroups are drawn. In this phase, the charts are used as aids to the practitioners in monitoring the process for any changes from an in-control state.

Another crucial matter is the subgroup size n of each rational subgroup. If n=1, then special care must be taken. Thus, four possibilities are considered:

— phase I and n=1, working with individual observations;

— phase I and n>1, working with rational subgroups;

— phase II and n=1, working with individual observations;

— phase II and n>1, working with rational subgroups.

6.2 Control charts for the process mean (n>1)

6.2.1 χ^2 control chart when pre-specified parameter values are known

Assume that the vector $\mathbf{x}$ follows a d-dimensional normal distribution, denoted as $N_d(\boldsymbol{\mu}_0,\boldsymbol{\Sigma}_0)$, and there are m subgroups each of size n>1 available from the process. Furthermore, assume that the observation vectors $\mathbf{x}$ are not time dependent. A control chart can be based on the sequence of the following statistic:

$$D_j^2 = n\left(\bar{\mathbf{x}}_j - \boldsymbol{\mu}_0\right)^{\mathrm{T}} \boldsymbol{\Sigma}_0^{-1}\left(\bar{\mathbf{x}}_j - \boldsymbol{\mu}_0\right) \qquad j = 1,2,\ldots, m \tag{1}$$

Here $\bar{\mathbf{x}}_j$ is the vector of the mean of the j-th rational subgroup, where $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are the known vector of means and the known variance-covariance matrix, respectively.

The D_j^2 statistic represents the weighted distance (Mahalanobis distance) of any point from the target $\boldsymbol{\mu}_0$. If the value of the test statistic D_j^2 plots above upper control limit, the chart signals a potential out-of-control process. In general, control charts have both upper and lower control limits. However, in this case only an upper control limit is used, because extreme values of the statistic correspond to points far remote from the target $\boldsymbol{\mu}_0$, whereas small or zero values of the statistic correspond to points close to the target $\boldsymbol{\mu}_0$.

The D_j^2 statistic follows a χ^2-distribution with d degrees of freedom. Thus, a multivariate Shewhart control chart for process mean, with known mean vector $\boldsymbol{\mu}_0$ and variance-covariance matrix $\boldsymbol{\Sigma}_0$, has the following upper control limit:

$$U_{\mathrm{CL}} = \chi^2_{1-\alpha,d} \tag{2}$$

For the determination of the upper control limit α can be chosen as 0,1 %, 0,2 %, 0,5 %, or 1 % under the consideration of a practical application. For example, the selection of 0,2 % means that there is approximately a 0,2 % risk of a false alarm, or an average of twice in a thousand, of a plotted point corresponding to the D_j^2 statistic being outside of the upper control limit when the process is in-control.

This control chart is called a phase II χ^2control chart.

6.2.2 T^2 control chart when pre-specified parameter values are unknown

When more than 20 subgroups are already obtained, and multivariate control charts are used to monitor the process, the sample mean vector $\bar{\bar{\mathbf{x}}}$ is estimated by the average of all subgroup means. The sample variance-covariance matrix $\bar{\mathbf{S}}$ is estimated by the $d \times d$ average of subgroup variance-covariance matrices. See Annex C.1.

If $\boldsymbol{\mu}_0$ is replaced $\bar{\bar{\mathbf{x}}}$, and $\boldsymbol{\Sigma}_0$ is replaced by $\bar{\mathbf{S}}$, with n>1 and $\bar{\mathbf{x}}_j$ is the mean of the j-th rational subgroup, a control chart can be based on the sequence of the following statistic[4]:

$$T_j^2 = n(\bar{\mathbf{x}}_j - \bar{\bar{\mathbf{x}}})^{\mathrm{T}} \bar{\mathbf{S}}^{-1} (\bar{\mathbf{x}}_j - \bar{\bar{\mathbf{x}}}) \qquad j = 1, 2, \ldots, m \tag{3}$$

for the j-th subgroup. Then the $T_j^2 / c_0(d,m,n)$ statistic follows an F distribution with d and $(mn-m-d+1)$ degrees of freedom. Here, $c_0(d,m,n) = [d(m-1)(n-1)](mn-m-d+1)^{-1}$.

Thus, a multivariate Shewhart control chart for process mean with unknown parameters has the following upper control limit:

$$U_{\mathrm{CL}} = \frac{d(m-1)(n-1)}{mn-m-d+1} F_{1-\alpha, d, mn-m-d+1} \tag{4}$$

This control chart is called a phase I T^2-chart.

If $\boldsymbol{\mu}_0$ is replaced by $\bar{\bar{\mathbf{x}}}$, and $\boldsymbol{\Sigma}_0$ is replaced by $\bar{\mathbf{S}}$, with n>1 and $\bar{\mathbf{x}}_f$ is the mean of a future rational subgroup, a control chart can be based on the sequence of the following statistic:

$$T_f^2 = n(\bar{\mathbf{x}}_f - \bar{\bar{\mathbf{x}}})^{\mathrm{T}} \bar{\mathbf{S}}^{-1} (\bar{\mathbf{x}}_f - \bar{\bar{\mathbf{x}}}) \tag{5}$$

Then the $T_f^2 / c_1(d,m,n)$ statistic follows an F-distribution with d and $(mn-m-d+1)$ degrees of freedom, where $c_1(d,m,n) = [d(m+1)(n-1)](mn-m-d+1)^{-1}$ and m is used to show the number of subgroups that belong to phase I.

Thus, a multivariate Shewhart control chart for process mean with unknown parameters, has the following upper control limit:

$$U_{\mathrm{CL}} = \frac{d(m+1)(n-1)}{mn-m-d+1} F_{1-\alpha, d, mn-m-d+1} \tag{6}$$

This control chart is called a phase II T^2-chart.

6.3 Control charts for the process mean (n=1)

6.3.1 χ^2 control chart when pre-specified parameter values are known

For charts constructed using individual observations (n=1), a control chart can be based on the sequence of the following statistic:

$$D_j^2 = \left(\mathbf{x}_j - \boldsymbol{\mu}_0\right)^{\mathrm{T}} \boldsymbol{\Sigma}_0^{-1} \left(\mathbf{x}_j - \boldsymbol{\mu}_0\right) \qquad j = 1, 2, \ldots, m \tag{7}$$

where $\mathbf{x}_j$ is the j-th, j =1, 2, ..., m, observation following $\mathrm{N}_d(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, where $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are the known vector of means and the known variance-covariance matrix, respectively. Moreover, assume that the observations $\mathbf{x}_j$ are not time dependent. The D_j^2 statistic follows a χ^2-distribution with d degrees of freedom. Thus, a multivariate Shewhart control chart for process mean with known mean vector $\boldsymbol{\mu}_0$ and variance-covariance matrix $\boldsymbol{\Sigma}_0$ has the following upper control limit:

$$U_{\mathrm{CL}} = \chi^2_{1-\alpha,d} \tag{8}$$

This control chart is called a phase II χ^2control chart.

6.3.2 T^2 control chart when pre-specified parameter values are unknown

When more than 20 observed vectors on multivariate characteristics are already obtained and multivariate control charts are used to monitor the process, the sample mean vector $\bar{\mathbf{x}}$ and the sample variance-covariance vector $\mathbf{S}$ are estimated. See Annex C.2.

If $\boldsymbol{\mu}_0$ is replaced by $\bar{\mathbf{x}}$, $\boldsymbol{\Sigma}_0$ is replaced by $\mathbf{S}$, and $\boldsymbol{x}_j$ is the j-th individual observation which is not independent of the estimators $\bar{\mathbf{x}}$ and $\mathbf{S}$, a control chart can be based on the sequence of the following statistic:

$$T_j^2 = (\mathbf{x}_j - \bar{\mathbf{x}})^{\mathrm{T}} \mathbf{S}^{-1} (\mathbf{x}_j - \bar{\mathbf{x}}) \quad j = 1, 2, \ldots, m \tag{9}$$

then the $T_i^2 / d_0(m)$ statistic follows beta distribution with $d/2$ and $\frac{1}{2}\left(\frac{2(m-1)^2}{3m-4} - d - 1\right)$ degrees of freedom, where $d_0(m) = (m-1)^2 m^{-1}$.

Thus, a multivariate Shewhart control chart for process mean with unknown parameters has the following upper control limit:

$$U_{\mathrm{CL}} = \frac{(m-1)^2}{m} B_{1-\alpha, d/2, \left(\frac{2(m-1)^2}{3m-4} - d - 1\right)/2} \tag{10}$$

This control chart is called a phase I T^2-chart.

If $\boldsymbol{\mu}_0$ is replaced by $\bar{\mathbf{x}}$, $\boldsymbol{\Sigma}_0$ is replaced by $\mathbf{S}$, and $\mathbf{x}_f$ is a future individual observation which is independent of the estimators $\bar{\mathbf{x}}$ and $\mathbf{S}$, a control chart can be based on the sequence of the following statistic:

$$T_f^2 = (\mathbf{x}_f - \bar{\mathbf{x}})^{\mathrm{T}} \mathbf{S}^{-1} (\mathbf{x}_f - \bar{\mathbf{x}}) \tag{11}$$

the $T_f^2 / d_1(m,d)$ statistic follows an F distribution with d and $(m-d)$ degrees of freedom, where $d_1(m,d) = d(m+1)(m-1)[m(m-d)]^{-1}$ and m is used to show the number of observed vectors that belong to phase I.

Thus, a multivariate Shewhart control chart for process mean with unknown parameters has the following upper control limit:

$$U_{\mathrm{CL}} = \frac{d(m+1)(m-1)}{m(m-d)} F_{1-\alpha,d,m-d} \tag{12}$$

This control chart is called a phase II T^2-chart.

6.4 Summary and selection of multivariate control charts with unweighted averages for mean shifts

Table 1 summarizes the statistic and upper control limits under different stati of multivariate control charts with unweighted averagse for mean shifts, and Figure 3 shows how to select those control charts.

Table 1 — Summary of multivariate control charts with unweighted averages

Status		Statistic	UCL	Title
n>1	parameter known	$D_j^2 = n\left(\bar{\mathbf{x}}_j - \mu_0\right)^{\mathrm{T}} \Sigma_0^{-1}\left(\bar{\mathbf{x}}_j - \mu_0\right)$	$\chi^2_{1-\alpha,d}$	phase II χ^2 chart with n>1
	phase I	$T_j^2 = n\left(\bar{\mathbf{x}}_j - \bar{\bar{\mathbf{x}}}\right)^{\mathrm{T}} \bar{\mathbf{S}}^{-1}\left(\bar{\mathbf{x}}_j - \bar{\bar{\mathbf{x}}}\right)$	$\dfrac{d(m-1)(n-1)}{mn-m-d+1} F_{1-\alpha,d,mn-m-d+1}$	phase I T^2 chart with n>1
	phase II	$T_f^2 = n\left(\bar{\mathbf{x}}_f - \bar{\bar{\mathbf{x}}}\right)^{\mathrm{T}} \bar{\mathbf{S}}^{-1}\left(\bar{\mathbf{x}}_f - \bar{\bar{\mathbf{x}}}\right)$	$\dfrac{d(m+1)(n-1)}{mn-m-d+1} F_{1-\alpha,d,mn-m-d+1}$	phase II T^2 chart with n>1
n=1	parameter known	$D_j^2 = \left(\mathbf{x}_j - \mu_0\right)^{\mathrm{T}} \Sigma_0^{-1}\left(\mathbf{x}_j - \mu_0\right)$	$\chi^2_{1-\alpha,d}$	phase II χ^2 chart with n=1
	phase I	$T_j^2 = \left(\mathbf{x}_j - \bar{\mathbf{x}}\right)^{\mathrm{T}} \mathbf{S}^{-1}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)$	$U_{\mathrm{CL}} = \dfrac{(m-1)^2}{m} B_{1-\alpha,d/2,\left(\frac{2(m-1)^2}{3m-4}-d-1\right)/2}$	phase I T^2 chart with n=1
	phase II	$T_f^2 = \left(\mathbf{x}_f - \bar{\mathbf{x}}\right)^{\mathrm{T}} \mathbf{S}^{-1}\left(\mathbf{x}_f - \bar{\mathbf{x}}\right)$	$\dfrac{d(m+1)(m-1)}{m(m-d)} F_{1-\alpha,d,m-d}$	phase II T^2 chart with n=1

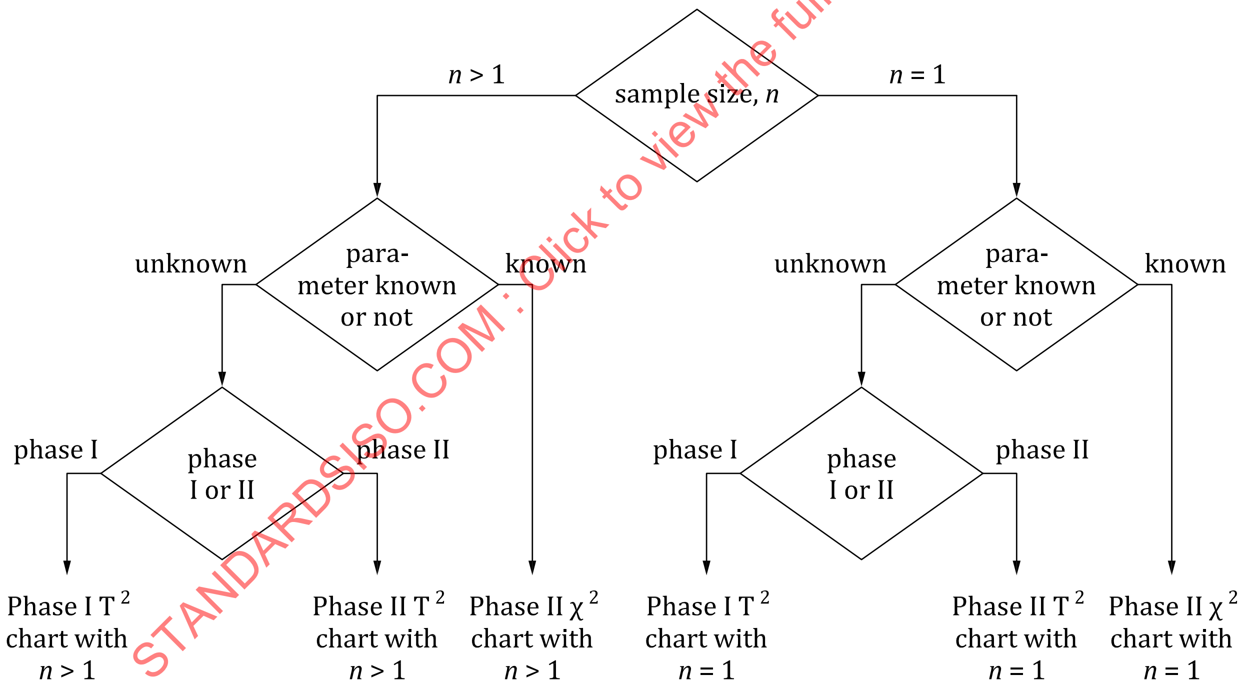


Figure 3 — Control charts selection flow chart

In many situations, a large number of preliminary samples would be required before the exact phase II control limits are well approximated. The recommended values of m are always greater than 20 preliminary samples, and often more than 50 samples especially for subgroup of size more than 10.

6.5 Test for assignable causes

In Shewhart $\bar{X}$ and X charts, a set of pattern tests is often used, like runs and trends, to indicate smaller shifts that cannot be large enough to manifest themselves quickly as points outside the control limits. As the statistic for χ^2 and T^2 control charts indicates the squared distance corresponding to points far remote from the target $\boldsymbol{\mu}_0$, it is not analogous to the points located above the central line or below the central line as in Shewhart $\bar{X}$ and X charts. For χ^2 and T^2 control charts, lower control limit does not exist. Thus, runs and trends in χ^2 and T^2 control charts cannot provide smaller shifts, like Shewhart $\bar{X}$ and X charts. For χ^2 and T^2 control charts, only if the value of the test statistic plots above upper control limit, the chart signals a potential out-of-control process.

To test smaller mean shifts of multivariate characteristics, multivariate control charts with weighted averages are applied.

7 Multivariate control charts with weighted averages for mean shifts

The χ^2 and T^2 charts described in Clause 6 are Shewhart-type control charts. They use information only from the current sample, so they are relatively insensitive to small and moderate shifts in the mean vector. The T^2 charts can be used in both phase I and II situations. EWMA control charts were developed to provide more sensitivity to small shifts in the univariate case, and they can be extended to multivariate quality control problems. As in the univariate case, the multivariate version of these charts is always used as a phase II procedure.

The MEWMA is a logical extension of the univariate EWMA, and is defined as follows[5]:

$$\mathbf{Z}_j = \boldsymbol{\lambda}\mathbf{x}_j + (\mathbf{I}-\boldsymbol{\lambda})\mathbf{Z}_{j-1} \qquad j = 1, 2, \ldots \tag{13}$$

where $\mathbf{Z}_0 = \boldsymbol{\mu}_0$, the in-control mean, and

$$\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d) \qquad 0 < \lambda_i \le 1 \tag{14}$$

It is usually assumed that λ_i are the same and $\lambda_i = \lambda$. The MEWMA can be written as:

$$\mathbf{Z}_j = \lambda\mathbf{x}_j + (1-\lambda)\mathbf{Z}_{j-1} \qquad j = 1, 2, \ldots \tag{15}$$

The constant λ, $0 < \lambda \le 1$, is the EWMA moving parameter. The MEWMA statistic reduces to $\mathbf{x}_i$ when $\lambda = 1$.

The MEWMA chart gives an out-of-control signal when:

$$Y_j^2 = \left(\mathbf{Z}_j - \boldsymbol{\mu}_0\right)^{\mathrm{T}} \Sigma_{\mathrm{Z}_j}^{-1} \left(\mathbf{Z}_j - \boldsymbol{\mu}_0\right) > h \tag{16}$$

where h represents the upper control limit of MEWMA, and the variance-covariance matrix Σ_{Z_j} of MEWMA statistic $\mathbf{Z}_j$ is:

$$\Sigma_{\mathrm{Z}_j} = \frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2j}\right]\Sigma_0 \tag{17}$$

which is analogous to the univariate EWMA. When $\lambda = 1$, the MEWMA chart reduces to a multivariate Shewhart-type control chart. If the mean vector $\boldsymbol{\mu}_0$ and the variance-covariance matrix $\boldsymbol{\Sigma}_0$ of $\mathbf{x}_j$ are known, the MEWMA chart reduces to the χ^2 chart when $\lambda = 1$.

An ARL analysis of the MEWMA control chart can be used to select λ and h based on simulation, under the consideration of subgroup size, shift size, given in-control ARL_0, and the ARL performance corresponding to a given shift size $\delta = \left[\left(\boldsymbol{\mu}-\boldsymbol{\mu}_0\right)^{\mathrm{T}} \Sigma^{-1} \left(\boldsymbol{\mu}-\boldsymbol{\mu}_0\right)\right]^{1/2}$. h is chosen to achieve a specified in-control ARL_0. λ is chosen to minimize the ARL for a shift size of the mean vector shown as following, and as given in Table 2.

— λ less than 0,05 is preferred to detect a shift of magnitude $\delta = 0{,}5$. Although the number of variables is important, a $\lambda = 0{,}03$ might be a good choice.

— For a shift of magnitude of approximately δ = 1, λ = 0,1 is close to the best choice over a wide range for d.

— For shifts from δ = 1,5 to 2, λ values from 0,15 to 0,25 are good choices.

— For shifts as large as δ = 3, λ = 0,4 is reasonable.

Table 2 — Determination of λ based on a shift size δ

Shift size δ	0,5	1	1,5 to 2	3
λ	0,03	0,1	0,15 to 0,25	0,4

NOTE Table 2 gives the determination of λ based on shift size δ. However, the exact shift size is a little easy to be determined in advance. Moreover, it is reasonable to monitor the process small shift instead of a level of unique shift size. ISO 7870-6 recommends that the most commonly used values of λ be between 0,25 and 0,5 inclusive.

See Annex B for the application of MEWMA control chart.

8 Control charts for the process dispersion

Process variability is summarized by the $d \times d$ variance-covariance matrix $\boldsymbol{\Sigma}$. The main diagonal elements of this matrix are the variances of the individual process variables, and the off-diagonal elements are the covariance. A direct procedure for monitoring the process dispersion is an extension of the univariate control chart, which is equivalent to repeated tests of significance of the hypothesis that the process variance-covariance matrix is equal to a particular matrix of constant $\boldsymbol{\Sigma_0}$. The statistic plotted on the control chart for the j-th subgroup is[6]:

$$W_j = -dn + dn\ln(n) - n\ln\left(\left|\mathbf{A}_j\right| / |\Sigma|\right) + \mathrm{tr}\left(\Sigma^{-1}\mathbf{A}_j\right) \tag{18}$$

where $\mathbf{A}_j = (n - 1)\mathbf{S}_j$, $\mathbf{S}_j$ is the variance-covariance matrix for j-th subgroup, and this tr is the trace operator. The trace of a matrix is the sum of the main diagonal elements. If the value of W_j plots above the upper control limit $U_{\mathrm{CL}} = \chi^2_{1-\alpha, d(d+1)/2}$, the process is out-of-control.

Another approach for monitoring the process dispersion is based on the sample *generalized* variance, $|\mathbf{S}|$, which is the determinant of the sample variance-covariance matrix $\mathbf{S}$. For example, one method is to use the mean and variance of $|\mathbf{S}|$, that is, $E(|\mathbf{S}|)$ and $V(|\mathbf{S}|)$, and the property that most of the probability distribution of $|\mathbf{S}|$ is contained in the interval $E(|\mathbf{S}|) \pm 3\sqrt{V(|\mathbf{S}|)}$, to determine the upper control limit and lower control limit for monitoring the process dispersion. Annex A provides an example which includes the application of control chart based on generalized variance for individual observations (n=1).

The multivariate extension of control charts for process dispersion is not as straightforward as that for the process mean. A general model and techniques that would encompass a wide range of problems encountered in practice is not available. In most cases, particular problems need to be handled in a specific manner.

Various alternative techniques for monitoring process dispersion have been introduced and have gained wide acceptance in practice. While extension of these techniques to the multivariate case is of great importance in practice, control procedures for monitoring the correlations among characteristics of a multivariate process have received very little attention. Perhaps this lack of progress is due to the fact that statistical inferences on the variance-covariance matrix are generally rather involved. Furthermore, unlike the problem of controlling process mean, it is not easy to define uniquely the shifts in the variance-covariance matrix that need to be detected. Another difficulty in designing a multivariate control procedure for dispersions is the identification of the out-of-control process parameter(s) when the control charts signals 'out-of-control'. In the multivariate case, the necessity for monitoring process variability is even more pronounced.

See Annex A for the application of multivariate control charts for process mean and process dispersion.

9 Interpretation of an out-of-control signal

Multivariate control charts are able to recognize an out-of-control process.

If a univariate control chart gives an out-of-control signal, then someone can easily detect the problem and find a solution since a univariate chart is associated with a single variable. This is not valid for a multivariate control chart, as a number of variables are involved and, also, correlations exist among them.

The identification of an out-of-control variable, or variables, after a multivariate control chart signals has been an interesting topic for academic research. An obvious idea is to consult the corresponding univariate control charts. Sometimes decomposition techniques are applied to identify particular subsets that cause an out-of-control signal. Principal components can also be used to investigate which of the variables are responsible for the creation of an out-of-control signal. However, the problem of interpreting an out-of-control signal is an open one which needs further investigation.

Annex A
(informative)

Example of multivariate statistical process control

A.1 General

For evaluating the quality of sonic welding, the depth of welding dissolution is monitored. In order to analyse the correlated characteristics which have an influence on the depth of welding dissolution, a multivariate control chart is applied. According to previous experience, three characteristics have been identified including the depth of welding ring, the insertion depth and the diameter of the horn mouth.

A.2 Observed data

Every month, one welding part is selected and analysed. The subgroup size is 1. The observed data for the depth of the welding ring, and the insertion depth and the diameter of the horn mouth are collected as given in Table A.1. They are used for the analysis of whether the process was in-control (phase I). A correlation matrix based on the observed data is obtained as given in Table A.2.

Table A.1 — Raw data of welding parts

No.	Depth of welding ring mm	Insertion depth mm	Diameter of horn mouth mm
1	10	19,4	31
2	8	19,2	31
3	10	19,96	30,5
4	10	18,8	31
5	8	18,52	26,5
6	5	20,4	26
7	6	20,84	29
8	10	19,44	30
9	10	20,36	32
10	10	20,28	33
11	10	20,24	32
12	4	19,08	29,5
13	10	20,52	32,5
14	10	20,56	31,5
15	4	20,12	30,5
16	3	17,96	28
17	6	19,08	30,5
18	10	18,8	31
19	5	20,68	32,5
20	6	19,64	30
21	6	18,52	27,5
22	4	18,16	32,5
23	6	18,04	29

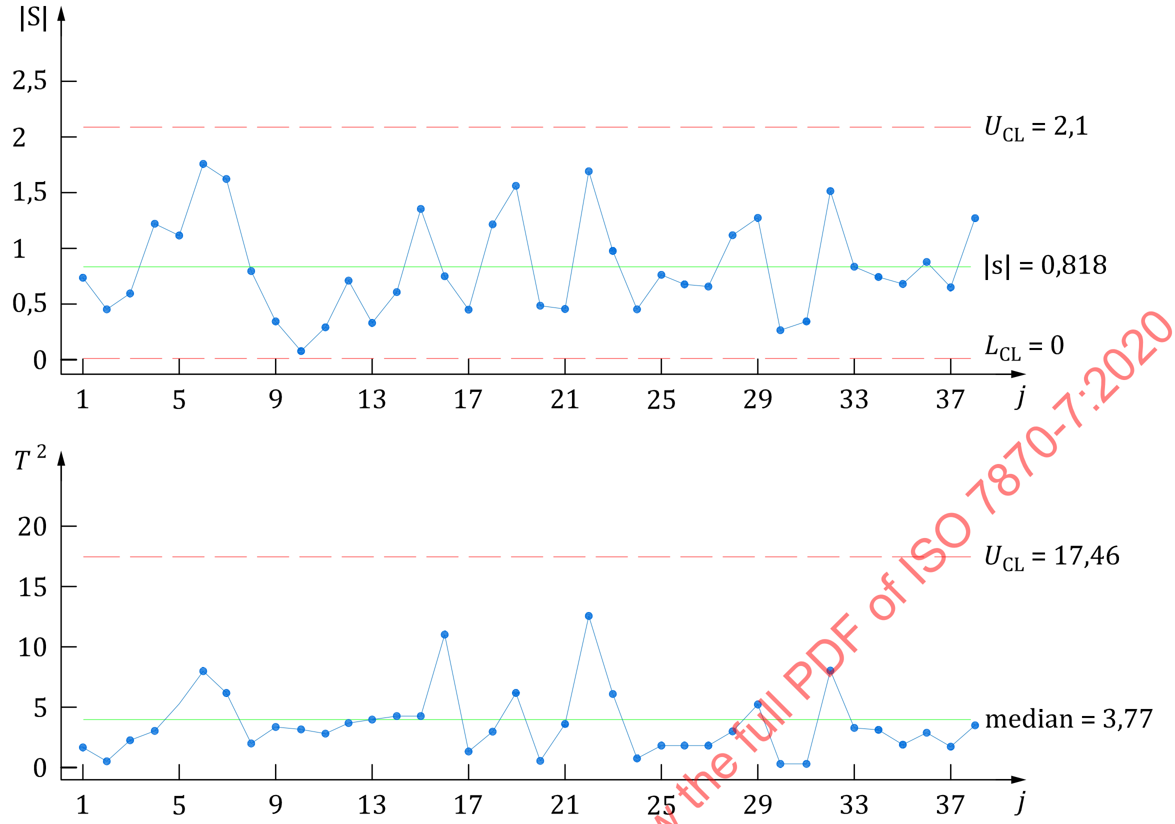
Table A.1 *(continued)*

No.	Depth of welding ring mm	Insertion depth mm	Diameter of horn mouth mm
24	8	19,28	29
25	8	19,88	28,5
26	8	19,2	28
27	10	19,52	32,5
28	10	19	32,5
29	8	19,88	26,5
30	8	19,4	31
31	8	19,32	31
32	6	20	35,5
33	6	19,44	26,5
34	10	19,64	34
35	5	19,4	30,5
36	5	19,4	31,5
37	10	19,52	32,5
38	5	20,32	29,5

Table A.2 — Correlation matrix

Characteristic	Depth of welding ring mm	Insertion depth mm	Diameter of horn mouth mm
Depth	1		
Insertion depth	0,201	1	
Diameter	0,342	0,227	1

A.3 Multivariate analysis



Key

j ongoing subgroup number

|S| generalized variance

T^2 Hotelling's T^2

Figure A.1 — T^2 control chart and control chart based on generalized variance

From Figure A.1, all points are located inside the control limits. No matter for T^2 control chart and control charts for the process dispersion, then it is in statistical control.

Annex B
(informative)

Example of MEWMA control chart

B.1 General

In the assembly process of electronic products, the quality and reliability of solder joints determine the quality of electronic products to a great extent. The monitoring data of solder joint quality is selected, including speed and temperature.

B.2 Observed data

Monitoring data using automatic data acquisition equipment is collected. 125 paired data are observed continuously as given in Table B.1. Then a correlation matrix is obtained as given in Table B.2.

Table B.1 — Raw data

No.	Speed m/min	Temperature °C
1	2,24	784
2	2,08	810
3	1,76	789
4	2,20	715
5	2,04	819
6	2,02	866
7	2,06	795
8	1,94	790
9	1,96	814
10	2,02	730
11	2,03	811
12	1,70	787
13	2,15	808
14	1,92	794
15	2,32	868
16	2,06	806
17	1,80	832
18	1,94	755
19	2,07	848
20	2,06	791
21	1,86	814
22	2,05	743
23	2,06	831
24	1,81	791
25	1,95	791
26	2,07	796

Table B.1 *(continued)*

No.	Speed m/min	Temperature °C
27	2,03	791
28	2,10	688
29	2,00	802
30	2,09	856
31	1,87	727
32	2,02	807
33	2,13	780
34	2,14	843
35	1,96	808
36	2,08	797
37	1,95	780
38	2,29	824
39	2,02	788
40	2,03	814
41	2,38	834
42	2,11	814
43	2,03	815
44	1,90	813
45	2,15	812
46	2,14	777
47	1,82	766
48	2,03	796
49	1,97	810
50	1,99	777
51	2,18	796
52	2,07	738
53	1,97	785
54	1,91	762
55	1,96	790
56	2,08	759
57	2,08	809
58	1,91	878
59	1,96	808
60	1,87	827
61	2,05	793
62	1,93	788
63	1,95	789
64	2,10	808
65	2,09	828
66	2,06	777
67	2,07	841
68	2,14	812
69	1,85	794

Table B.1 *(continued)*

No.	Speed m/min	Temperature °C
70	1,89	821
71	1,99	801
72	2,09	765
73	1,97	795
74	2,01	796
75	2,04	804
76	1,97	814
77	2,09	780
78	1,96	766
79	2,04	826
80	1,96	795
81	1,90	799
82	1,97	805
83	1,87	825
84	2,08	798
85	2,14	840
86	1,88	765
87	1,89	716
88	1,98	755
89	1,97	799
90	1,90	787
91	2,07	803
92	2,01	750
93	2,03	818
94	1,98	788
95	2,08	822
96	2,07	823
97	1,99	798
98	1,98	846
99	2,13	829
100	1,97	754
101	1,93	823
102	2,07	747
103	2,03	788
104	2,03	823
105	2,06	784
106	2,03	874
107	1,73	728
108	1,92	792
109	2,08	810
110	1,98	819
111	2,02	757
112	2,04	778